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Convergence of the Fuzzy Sequence of Successive Approximations to a Fixed Point

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Abstract
In this paper, the existence of the fixed point of fuzzy sequence of successive approximations is proved. For this idea, the $\tau -$distance is defined and some theorems are proved in detail.

Keywords: *Fuzzy Differential Equation, Fixed Point, Minimal and Maximal Solutions, Existence and Uniqueness of Solution.*

1 Introduction

The Banach contraction principle and fixed point theorem are forceful tools in nonlinear analysis, control theory, economic theory, and global analysis. These theorems are extended by several authors.

The theory of fuzzy functions and its application has increased recently due to the industrial interest of fuzzy control. Moreover, in view of the development of the calculus for fuzzy functions, the investigation of fuzzy differential equations has been initiated and the existence and uniqueness of solutions of fuzzy initial value problem was considered under a Lipschitz condition [1, 2]. The basic result such as existence, uniqueness, continuity with respect to initial values and global existence, are proved by developing the needed general comparison principle which helps to understand the intricacies involved in incorporating fuzziness in differential equations [3].

Recently, the above concept has been extended to the integro-differential equations by Balasubramaniam and Muralisankar, [4].

Existence of fixed points in partially ordered sets has been considered recently in [5, 6, 7]. In [8, 9] some fixed point theorems are proved for a mixed monotone mapping in a metric space endowed with a partial order and the authors apply their results to problems of existence and uniqueness of solutions for some boundary value problems.

2 Basic concept

A nonempty subset A of R is called convex if and only if $(1-k)x+ky \in A$ for every $x, y \in A$ and $k \in [0,1]$. By $p_k(R)$, we denote the family of all nonempty compact convex subsets of R.
There are various definitions for the concept of fuzzy numbers ([10, 11]).

Definition 2.1 A fuzzy number is a function $u: R \to [0,1]$ satisfying the following properties:

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(i) u is normal, i.e. $\exists x_0 \in R$ with $u(x_0)=1$,
(ii) u is a convex fuzzy set (i.e. $u(\lambda x+(1-\lambda)y)\geq \min\{u(x),u(y)\}\forall x,y\in R,\lambda\in[0,1]$),
(iii) u is upper semi-continuous on R,
(iv) $\overline{\{x\in R: u(x)>0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

The set of all fuzzy real numbers is denoted by E. Obviously $R\subset E$. Here $R\subset E$ is understood as $R=\{\ \chi_x : \chi$ is usual real number$\}$. For $0<r\leq 1$, denote $[u]_r=\{x\in R; u(x)\geq r\}$ and $[u]_0=\overline{\{x\in R; u(x)>0\}}$. Then it is well-known that for any $r\in[0,1]$, $[u]_r$ is a bounded closed interval. For $u,v\in E$, and $\lambda\in R$, where sum $u+v$ and the product $\lambda.u$ are defined by $[u+v]_r=[u]_r+[v]_r$, $[\lambda.u]_r=\lambda[u]_r$, $\forall r\in[0,1]$, where $[u]_r+[v]_r=\{x+y: x\in[u]_r, y\in[v]_r\}$ means the conventional addition of two intervals (subsets) of R and $\lambda[u]_r=\{\lambda x: x\in[u]_r\}$ means the conventional product between a scalar and a subset of R (see e.g. [10, 12].
Another definition for a fuzzy number is as follows:

Definition 2.2 An arbitrary fuzzy number in the parametric form is represented by an ordered pair of functions $(\underline{u}(r),\overline{u}(r))$, $0\leq r\leq 1$, which satisfy the following requirements:

- $\underline{u}(r)$ is a bounded left-continuous non-decreasing function over $[0,1]$.
- $\overline{u}(r)$ is a bounded left-continuous non-increasing function over $[0,1]$.
- $\underline{u}(r)\leq\overline{u}(r)$, $0\leq r\leq 1$.

A crisp number α is simply represented by $\underline{u}(r)=\overline{u}(r)=\alpha$, $0\leq r\leq 1$. We recall that for $a<b<c$, $a,b,c\in R$, the triangular fuzzy number $u=(a,b,c)$ determined by a,b,c is given such that $\underline{u}(r)=a+(b-c)r$ and $\overline{u}(r)=c-(c-b)r$ are the endpoints of the r-level sets, for all $r\in[0,1]$. Here $\underline{u}(r)=\overline{u}(r)=b$ and it is denoted by $[u]_1$. For arbitrary $u=(\underline{u}(r),\overline{u}(r))$, $v=(\underline{v}(r),\overline{v}(r))$ we define addition and multiplication by k as

- $\underline{(u+v)}(r)=(\underline{u}(r)+\underline{v}(r))$,
- $\overline{(u+v)}(r)=(\overline{u}(r)+\overline{v}(r))$,
- $\underline{(ku)}(r)=k\underline{u}(r), \overline{(ku)}(r)=k\overline{u}(r), \qquad k\geq 0$,
- $\underline{(ku)}(r)=k\overline{u}(r), \overline{(ku)}(r)=k\underline{u}(r), \qquad k<0$.

In this paper, we represent an arbitrary fuzzy number with compact support by a pair of functions $(\underline{u}(r),\overline{u}(r)), 0\leq r\leq 1$. Also, we use the Hausdorff distance between fuzzy numbers. This fuzzy number space as shown in [13] can be embedded into Banach space $B=\overline{c}[0,1]\times\overline{c}[0,1]$ where the metric is usually defined as follows: Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded $r-$level sets. Since the $r-$cuts of fuzzy numbers are always closed and bounded, the intervals are written as

$u[r]=[\underline{u}(r),\overline{u}(r)]$, for all r. We denote by ω the set of all nonempty compact subsets of R and by ω_c the subsets of ω consisting of nonempty convex compact sets. Recall that

$$\rho(x,A)=\min_{a\in A}\|x-a\|$$

is the distance of a point $x\in R$ from $A\in\omega$ and the *Hausdorff separation* $\rho(A,B)$ of $A,B\in\omega$ is defined as

$$\rho(A,B)=\max_{a\in A}\rho(a,B).$$

Note that the notation is consistent, since $\rho(a,B)=\rho(\{a\},B)$. Now, ρ is not a metric. In fact, $\rho(A,B)=0$ if and only if $A\subseteq B$. The *Hausdorff metric* d_H on ω is defined by

$$d_H(A,B)=\max\{\rho(A,B),\rho(B,A)\}.$$

The metric d_∞ is defined on E as

$$d_\infty(u,v)=\sup\{d_H(u[r],v[r]):0\le r\le 1\},\ u,v\in E.$$

for arbitrary $(u,v)\in\bar{c}[0,1]\times\bar{c}[0,1]$. The following properties are well-known. (see e.g. [11, 12])

- $d_\infty(u+w,v+w)=d_\infty(u,v)$, $\quad\forall u,v,w\in E$,
- $d_\infty(k.u,k.v)=|k|d_\infty(u,v)$, $\quad\forall k\in R,u,v\in E$,
- $d_\infty(u+v,w+e)\le d_\infty(u,w)+d_\infty(v,e)$, $\quad\forall u,v,w,e\in E$,
- $d_\infty(u,v)=d_\infty(v,u)$, $\quad\forall u,v\in E$.

Theorem 2.1

- If we define $\widetilde{0}=\chi_0$, then $\widetilde{0}\in E$ is a neutral element with respect to addition, i.e. $u+\widetilde{0}=\widetilde{0}+u=u$, for all $u\in E$.
- With respect to $\widetilde{0}$, none of $u\in E\setminus R$, has opposite in E.
- For any $a,b\in R$ with $a,b\ge 0$ or $a,b\le 0$ and any $u\in E$, we have $(a+b).u=a.u+b.u$; however, this relation dose not necessarily hold for any $a,b\in R$, in general.
- For any $\lambda\in R$ and any $u,v\in E$, we have $\lambda.(u+v)=\lambda.u+\lambda.v$.
- For any $\lambda,\mu\in R$ and any $u\in E$, we have $\lambda.(\mu u)=(\lambda.\mu)u$.(see [12])

Definition 2.3 Let E be a set of all fuzzy numbers, we say that $f(x)$ is a fuzzy function if $f: E \to E$.

Definition 2.4 The fuzzy function $f: E \to E$ is continuous if for arbitrary fixed $x_0 \in E$ and $\varepsilon > 0$, exists $\delta > 0$ such that

$$d_\infty(x, x_0) < \delta \quad \Rightarrow \quad d_\infty(f(x), f(x_0)) < \varepsilon$$

Definition 2.5 Let $f: E \to E$ is a fuzzy function. Then f is non-decreasing fuzzy function if for arbitrary fuzzy numbers x_1 and x_2 we have
$$d_\infty(x_1, \widetilde{0}) \le d_\infty(x_2, \widetilde{0}) \Rightarrow d_\infty(f(x_1), \widetilde{0}) \le d_\infty(f(x_2), \widetilde{0})$$

or

$$x_1 {}^\circ x_2 \Rightarrow f(x_1)^\circ f(x_2)$$

Definition 2.6 For arbitrary fuzzy numbers u and v, define the ranking of u and v by the Hausdorff metric on E, i.e.,

- $u \,\mathring{}\, v$ means that $d_\infty(u, \widetilde{0}) \le d_\infty(v, \widetilde{0})$
- $u \pm v$ means that $d_\infty(u, \widetilde{0}) \ge d_\infty(v, \widetilde{0})$
- $u : v$ means that $d_\infty(u, \widetilde{0}) = d_\infty(v, \widetilde{0})$

3 Main results

In the this section, we are going to study the uniqueness and existence of solutions to the periodic first-order fuzzy differential equation with boundary value.

Definition 3.1 Let (E, d_∞) be a metric space. Then a function ρ from $E \times E$ into $[0, \infty)$ is called a fuzzy $\tau-$distance on E if there exists a function η from $E \times [0, \infty)$ into $[0, \infty)$ and the following are satisfied:

- $\rho(x, z) \le \rho(x, y) + \rho(y, z), \quad \forall x, y, z \in E$,
- $\eta(x, 0) = 0a$ and $\eta(x, t) \ge t$ for all $x \in E$ and $t \in [0, \infty)$ and η is concave and continuous in its second variable,
- $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, \rho(z_n, x_m)) \mid m \ge n\} = 0$ imply $\rho(\omega, x) \le \liminf_n \rho(\omega, x)$ for all $\omega \in E$,
- $\lim_n \sup\{\rho(x_n, y_m) \mid m \ge n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply that $\lim_n \eta(y_n, t_n) = 0$,
- $\lim_n \eta(z_n, \rho(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, \rho(z_n, y_n)) = 0$ imply that $\lim_n d_\infty(x_n, y_n) = 0$.

Lemma 3.1 Let (E, d_∞) be a complete metric space and let ρ be a $\tau-$distance on E. Assume that a sequence $\{x_n\}$ in E satisfies $\lim_n \sup\{\rho(x_n, x_m) \mid m > n\} = 0$,

$\lim_n \rho(x_n, y) = 0$, and $\lim_n \rho(x_n, z) = 0$. Then $d_\infty(y,z) = 0$.

Proof. Using Definition 3.1, the proof is clear.

Theorem 3.1 Let (E, d_∞) be a complete metric space and let T be a mapping on E. Assume that there exist a $\tau-$distance, $\rho, r \in [0,1)$, and $M \in [0,\infty)$ such that

$$\rho(Tx, T^2x) \le r\rho(x, Tx), \quad x \in E$$

Assume the following:

(i) If $\lim_n \sup\{\rho(x_n, x_m) \mid m > n\} = 0, \lim_n \rho(x_n, Tx_n) = 0,$ and $\lim_n \rho(x_n, y) = 0,$ then $d_\infty(Ty, y) = 0$

Then there exists $x_0 \in E$ such that $d_\infty(Tx_0, x_0) = 0$ and $\rho(x_0, x_0) = 0$.

Proof. We shal prove T has fixed point x_0 with $\rho(x_0, x_0) = 0$ in case (i). Fix $u \in E$ and put $u_n := T^n u$ for all $n \in N$. Then if $m > n$, we have

$$\rho(u_n, u_m) \le \sum_{k=n}^{m-1} \rho(u_k, u_{k+1}) \le \sum_{k=n}^{m-1} r^k \rho(u, u_1) \le \frac{r^n}{1-r} \rho(u, u_1)$$

and hence

$$\lim_n \sup\{\rho(u_n, u_m) \mid m > n\} = 0.$$

So, $\{u_n\}$ is a Cauchy sequence. Since (E, d_∞) be a complete metric space, $\{u_n\}$ converges to some point $x_0 \in E$. From third condition in Definition 3.1, we have

$$\begin{aligned}
\lim_{n\to\infty} \sup(\rho(u_n, Tu_n) + \rho(u_n, x_0)) &\le \lim_{n\to\infty} \sup\left(\rho(u_n, u_{n+1}) + \lim_{m\to\infty} \inf \rho(u_n, u_m)\right) * 3mm \\
&\le 2 \lim_{n\to\infty} \sup_{m>n} \rho(u_n, u_m) * 3mm \\
&= 0
\end{aligned}$$

therefore $d_\infty(Tx_0, x_0) = 0$ from (i). Further we obtain

$$\begin{aligned}
\rho(x_0, x_0) &= \rho(Tx_0, T^2x_0) * 3mm \\
&\le r\rho(x_0, Tx_0) * 3mm \\
&= r\rho(x_0, x_0)
\end{aligned}$$

thus, $\rho(x_0, x_0) = 0$.

Theorem 3.2 Let (E, d_∞) be a complete metric space and let T be a mapping on E. Assume that there exist a $\tau-$distance, $\rho, r \in [0,1)$, and $M \in [0,\infty)$ such that

$$\rho(Tx, T^2x) \le r\rho(x, Tx), \quad \rho(Tx, Ty) \le M\rho(x, y), \quad x, y \in E \tag{1}$$

Then

$$\{T^n x\} \text{ converges to a fixed point for every } x \in E. \qquad (2)$$

Moreover, if $d_\infty(Tz, z) = 0$, then $\rho(z, z) = 0$.

Proof. Assume that $\lim_n \sup\{\rho(x_n, x_m) \mid m > n\} = 0$, $\lim_n \rho(x_n, Tx_n) = 0$, and $\lim_n \rho(x_n, y) = 0$. Then we have

$$\rho(x_n, Ty) \leq \rho(x_n, Tx_n) + \rho(Tx_n, Ty) \leq \rho(x_n, Tx_n) + M\rho(x_n, y) \qquad (3)$$

and hence, $\lim_n \rho(x_n, Ty) = 0$. By Lemma 3.1, we obtain $d_\infty(Ty, y) = 0$. By Theorem 3.1, we obtain the desired result.

4 Conclusion

In this work, we studied the existence of the fixed point of fuzzy sequence of successive approximations. For this idea, we defined the $\tau-$distance and then by suing this definition, we proved some theorems in detail.

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