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Variational Iteration Method for Solving Nonlinear System of Fredholm Integro-Differential Equations

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Abstract
In this paper, we solve nonlinear system of Fredholm integro-differential equation that it is a class of important models in applied science. Since He's variational iteration method is very effective and convenient method such that it can be found analytic solution of the above equation in a closed form and avoids the round of errors for finding approximates, so we use variational iteration method for solving nonlinear system of integro-differential equations. Finally we will solve two example for illustrating validity and applicability of the (VIM) method.

Keywords: Iteration Method, Nonlinear System, Fredholm, Differential Equations.

1 Introduction

For solving integral equations types of numerical and analytical methods was proposed, such as [1, 2]. Also in [3] Fredholm integro-differential equations system is solved by Adomian decomposition method. In [4, 5, 6] variational iteration method is used to solve ordinary differential equations and nonlinear differential equations systems.

In [7, 8] nonlinear fluid flows and Fokker-Planck equations is solved by using (VIM) method. For finding convergence of (VIM) method see [9]. In general case we consider to nonlinear operator form as follows [10]:

$$L[u(t)] + N[u(t)] = g(t), \tag{1}$$

where L is a linear operator, N is a nonlinear operator, $g(x)$ is a known and $u(x)$ is a unknown function to be determined. By considering the iteration sequences (VIM) method is the following form:

$$u_{n+1}(x) = u_n(x) + \int_{\circ}^{t} \lambda(s)[Lu_n(s) + N\tilde{u}_n(s) - g(s)]ds, \tag{2}$$

where λ is a general Lagrange multipliers and optimal value it can be found via variational theory [11].

In order to by introducing $\tilde{u}_n(s)$ as a restrict variation and choose of $\delta\tilde{u}_n = \circ$ [11], we obtain optimal value of $\lambda(t,s)$. By substituting $\lambda(t,s)$ in (2) and using of $u_{\circ}(x)$ as an initial approximation, we will find a sequence of approximations to approximate of exact solution.

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2 Nonlinear System of Fredholm Integro-Differential Equations

We consider a general nonlinear system of Fredholm integro-differential equations,

$$\begin{cases} u_1^{(m)}(x) = H_1\left(x, u_2(x),...,u_2^{(m)}(x), u_3(x),...,u_3^{(m)}(x),..., u_n(x), ..., u_n^{(m)}(x)\right) \\ \qquad +\int_{\circ}^{1} k_1\left(x, t, u_1(t),...,u_1^{(m)}(t),..., u_n(t),...,u_n^{(m)}(t)\right) dt, \\ u_2^{(m)}(x) = H_2\left(x, u_1(x),...,u_1^{(m)}(x), u_3(x),...,u_3^{(m)}(x), ..., u_n(x),...,u_n^{(m)}(x)\right) \\ \qquad +\int_{\circ}^{1} k_2\left(x, t, u_1(t),...,u_1^{(m)}(t),..., u_n(t),...,u_n^{(m)}(t)\right) dt, \\ . \qquad . \qquad . \\ . \qquad . \qquad . \\ . \qquad . \qquad . \\ u_n^{(m)}(x) = H_n\left(x, u_1(x),...,u_1^{(m)}(x), u_2(x),...,u_2^{(m)}(x), ..., u_{n-1}(x),...,u_{n-1}^{(m)}(x)\right) \\ \qquad +\int_{\circ}^{1} k_n\left(x, t, u_1(t),...,u_1^{(m)}(t),..., u_n(t),...,u_n^{(m)}(t)\right) dt, \end{cases} \tag{3}$$

where m denotes mth order derivatives H_i and k_i for $i = 1,2,...,n$ are continuous functions and $u_i(x)$ for $i = 1,2,...,n$ are unknown functions. For solving (3) by variational iteration method we choose restrict variations as follows:

$$\begin{cases} u_{i,k+1}(x) = u_{i,k}(x) + \int_{\circ}^{x} \lambda_i(s)\Big[u_{i,k}^{(m)}(s) - H_i\left(s, \tilde{u}_{1,k}(s),...,\tilde{u}_{1,k}^{(m)}(s),...,\right. \\ \qquad \tilde{u}_{i-1,k}(s),...,\tilde{u}_{i-1,k}^{(m)}(s), \tilde{u}_{i+1,k}(s),...,\tilde{u}_{i+1,k}^{(m)}(s),..., \tilde{u}_{n,k}(s),...,\tilde{u}_{n,k}^{(m)}(s) \\ \qquad -\int_{\circ}^{s} k_i\left(s, t, \tilde{u}_{1,k}(t),...,\tilde{u}_{1,k}^{(m)}(t),..., \tilde{u}_{n,k}(t),...,\tilde{u}_{n,k}^{m}(t)\right) dt \Big] ds, \\ i = 1,2,...,n. \end{cases} \tag{4}$$

For finding optimal value of λ, we construct correction functional by effect δ on the both sides of (4), so we have

$$\delta u_{i,k+1}(x) = \delta u_{i,k}(x) + \delta \int_{\circ}^{x} \lambda_i(s)\Big[u_{i,k}^{(m)}(s) - H_i\left(s, \tilde{u}_{1,k}(s),...,\tilde{u}_{1,k}^{(m)}(s),...,\right.$$

$$\left.\tilde{u}_{i-1,k}(s),...,\tilde{u}_{i-1,k}^{(m)}(s), \tilde{u}_{i+1,k}(s),...,\tilde{u}_{i+1,k}^{(m)}(s),..., \tilde{u}_{n,k}(s),...,\tilde{u}_{n,k}^{(m)}(s)\right)$$

$$-\int_{\circ}^{1} k_i\left(s, t, \tilde{u}_{1,k}(t),...,\tilde{u}_{1,k}^{(m)}(t),....., \tilde{u}_{n,k}(t),...,\tilde{u}_{n,k}^{(m)}(t)\right) dt \Big] ds = \circ,$$

$i = 1,2,...,n$.

Because $\delta \tilde{u} = \circ$, so

$$\delta u_{i,k+1}(x) = \delta u_{i,k}(x) + \delta \int_{\circ}^{x} \lambda_i(s) u_{i,k}^{(m)}(s) \; ds = \circ.$$

By using integration by parts stationary conditions can be found as follows:

$$\delta u_{i,k+1}(x) = \delta u_{i,k}(x) + \lambda_i(s)\,\delta u_{i,k}^{(m-1)}(s)\big|_{s=x} - \lambda_i'(s)\,\delta u_{i,k}^{(m-2)}(s)\big|_{s=x} + \ldots$$
$$+ (-1)^{(m-1)}\lambda_i^{(m-1)}(s)\,\delta u_{i,k}(s)\big|_{s=x} + (-1)^m \int_0^x \lambda_i^{(m)}(s)\,\delta u_{i,k}(s)$$

$$= (1 + (-1)^{m-1}\lambda_i^{(m-1)})\ \delta u_{i,k}(x)\big|_{s=x} + \lambda_i(s)\,\delta u_{i,k}^{(m-1)}(s)\big|_{s=x} - \lambda_i'(s)\,\delta u_{i,k}^{(m-2)}(s)\big|_{s=x}$$
$$+ \ldots + (-1)^m \int_0^x \lambda_i^{(m)}(s)\,\delta u_{i,k}(s) = 0.$$

So

$$\begin{aligned}
&\lambda_i^{(m)}(s) = 0,\\
&1 + (-1)^{m-1}\lambda_i^{(m-1)}(s)\big|_{s=x} = 0,\\
&\lambda_i^{(j)}(s)\big|_{s=x} = 0,\ j = 1,2,\ldots,m-2,\quad i = 1,2,\ldots,n,\\
&\lambda_i(s)\big|_{s=x} = 0.
\end{aligned} \tag{5}$$

The Lagrange multipliers can be expressed as follows:

$$\lambda_i(s) = \frac{(-1)^m}{(m-1)!}(s-x)^{m-1}.$$

By substituting $\lambda_i(s)$ in (4) we obtain an iteration algebraic system for finding solution of (3) the following form:

$$\begin{aligned}
&u_{i,k+1}(x) = u_{i,k}(x) + \int_0^x \frac{(-1)^m}{(m-1)!}(s-x)^{m-1}\Big\{u_{i,k}^{(m)}(s) - H_i\Big[s, u_{1,k}(s),\ldots,u_{1,k}^{(m)}(s),\ldots,\\
&u_{i-1,k}(s),\ldots,u_{i-1,k}^{(m)}(s), u_{i+1,k}(s),\ldots,u_{i+1,k}^{(m)}(s),\ldots, u_{n,k}^{(m)}(s),\ldots,u_{n,k}^{(m)}(s)\Big]\\
&-\int_0^1 k_i\left(s, t, u_{1,k}(t),\ldots,u_{1,k}^{(m)}(t),\ldots, u_{n,k}(t),\ldots,u_{n,k}^{(m)}(t)\right) dt\Big\}ds,\quad i = 1,2,\ldots,n.
\end{aligned} \tag{6}$$

3 Application

In this section, we solve two examples of the nonlinear system of Fredholm integro-differential equations by using variational iteration method. In the second example we use another kind of $u_0(x)$ and $v_0(x)$ as an initial approximation.

Example1. Consider the following system of Fredholm integro-differential equations:

$$\begin{cases}
u''(x) = \dfrac{8}{9} + \displaystyle\int_0^1 x\left(\frac{u(t)}{3} + \frac{v(t)}{4}\right)dt, & u(0) = 0,\ u'(0) = \dfrac{1}{3},\\
v''(x) = 6x - \dfrac{x^2}{18} + \displaystyle\int_0^1 \left(\frac{u(t)}{6} - \frac{v(t)}{3}\right)dt, & v(0) = 0,\ v'(0) = \dfrac{1}{2},
\end{cases}$$

with exact solution $u(x)=\frac{x^2}{2}+\frac{1}{3}x$ and $v(x)=x^3-\frac{x}{2}$. By considering of the algebraic system (6) and the above initial conditions, we obtain $u_\circ(x)=-\frac{1}{3}x$ and $v_\circ(x)=-\frac{x}{2}$ as an initial approximation. By using $u_\circ(x)$ and $v_\circ(x)$ in the algebraic system (6) the numerical results in third iteration are shown in table1.

Table 1 Numerical results for example1

(x,y)	$\lvert u(x,y)-u_3(x,y)\rvert$	$\lvert v(x,y)-v_3(x,y)\rvert$
0	0	0
0·1	$5\cdot37528\times10^{-4}$	$2\cdot54287\times10^{-4}$
0·2	$2\cdot078\times10^{-3}$	$1\cdot01159\times10^{-3}$
0·3	$4\cdot51326\times10^{-3}$	$2\cdot25525\times10^{-3}$
0·4	$7\cdot73514\times10^{-3}$	$3\cdot95747\times10^{-3}$
0·5	$1\cdot16355\times10^{-2}$	$6\cdot07939\times10^{-3}$
0·6	$1\cdot61061\times10^{-2}$	$8\cdot57098\times10^{-3}$
0·7	$2\cdot10389\times10^{-2}$	$1\cdot13712\times10^{-2}$
0·8	$2\cdot63256\times10^{-2}$	$1\cdot44077\times10^{-2}$
0·9	$3\cdot18581\times10^{-2}$	$1\cdot75972\times10^{-2}$
1	$3\cdot75282\times10^{-2}$	$2\cdot08543\times10^{-2}$

Example2.
Consider nonlinear system of Fredholm integro-differential equations as follows:

$$\begin{cases} u'(x)=1-3x+\int_\circ^1 x\left[u^2(t)+2v(t)\right]dt, & u(0)=1,\ u'(0)=1, \\ v''(x)=2-\frac{25}{12}x+\int_\circ^1 x\left[u\;(t)+u(t)v(t)\right]dt, & v(0)=0,\ v'(0)=0, \end{cases}$$

with exact solution $u(x)=x+1$ and $v(x)=x^2$.
Because there are $u'(x)$ and $v''(x)$, then $u(x)$ and $v(x)$ are continuous, so solution of space is $C[0,1]$ for every equation of the above system.
Thus, we suggest $u(x)$ and $v(x)$ as solution of the above system as a linear combination of $1,x,x^2,\ldots$ functions in other words,
$u(x),v(x)\in$ Linear Span $\{1,x,x^2,\ldots\}$.
So, we use $u_\circ(x)=ax+b$ and $v_\circ(x)=dx^2+nx+m$ as an initial approximation. By considering initial conditions of example2 we have,
$1=u(0)=u_\circ(0)=b$,
$1=u'(0)=u'_\circ(0)=a$,

then $u_\circ(x) = x + 1$. Also
$0 = v(0) = v_\circ(0) = m$,
$0 = v'(0) = v'_\circ(0) = n$,
then $v_\circ(x) = dx^2$. According to system (6), we have an iteration system the following form:

$$\begin{cases} u_{n+1}(x) = u_n(x) - \int_\circ^x \left[u'(s) - 1 + 3s - \int_\circ^1 s[u^2(t) + 2v(t)]\,dt \right] ds, \\ v_{n+1}(x) = v_n(x) + \int_\circ^x (s-x) \left[v''(s) - 2 + \frac{25}{12}s - \int_\circ^1 s[u(t) + u(t)v(t)]\,dt \right] ds, \end{cases} \tag{7}$$

By substituting $u_\circ(x) = x + 1$ in (7), $u_1(x)$ and $v_1(x)$ can be expressed by

$$u_1(x) = (1+x) - \frac{x^2}{3}(1-d),$$
$$v_1(x) = dx^2 + \frac{25}{12}\left(1 - \frac{18+7d}{25}\right)\left(-\frac{x^3}{6}\right).$$

For simplification we choose $d = 1$ in $u_1(x)$ and $v_1(x)$, so

$$u_1(x) = u_\circ(x) = 1 + x,$$
$$v_1(x) = v_\circ(x) = x^2.$$

Thus $u_1(x)$ and $v_1(x)$ are equal to exact solution.

4 Conclusion

Nonlinear problems are important in applied sciences and engineering, so solution of these problems are important but in the general case difficult.
Fortunately variational iteration method is effective and easy for solving these problems, and in this paper we used it for solving nonlinear system of Fredholm integro-differential equations that the numerical results have a high accuracy such that in the example2 they are equal to exact solution.

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