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# Inverse Nodal Problem for Differential Pencils with Separated Boundary Conditions

**Sh. Akbarpoor*1, A. Dabbaghian2**
1- Islamic Azad University, Jouybar Branch, Jouybar, Iran
2- Islamic Azad University, Neka Branch, Neka, Iran

**Abstract**
In this paper, uniqueness theorem is studied for the diffusion operator on a finite interval with separated boundary conditions. The oscillation of the eigenfunctions corresponding to large modulus eigenvalues is established and an asymptotic of the nodal points is obtained. By using these new spectral parameters, uniqueness theorem is proved.

**Keywords:** Differential Pencils, Eigenvalues, Eigenfunction, Nodal Points, Inverse Problem.

**MSC:** 34B24-34B05

## 1 Introduction

Differential equations with nonlinear dependence on the spectral parameter arise in various problems of mathematics as well as in applications, for example, the diffusion operator (see [1,2] for details). Inverse problems of spectral analysis consist in recovering operators from their spectral characteristics. For differential pencils the inverse problem were studied in ([3-6]) and other works. We consider the problem

$$Ly(x) := y''(x) + (\rho^2 - 2\rho q_1(x) - q_0(x))y(x) = 0, \qquad 0 < x < 1, \tag{1}$$

$$U(y) := y'(0) - hy(0) = 0, \qquad V(y) := y'(1) + Hy(1) = 0. \tag{2}$$

Here $\rho$ is the spectral parameter. Let $q_0(x), q_1(x)$ are real functions, $h, H$ are real numbers and $q_0(x), q_1(x) \in W_1^2[0,1]$.

In recent years, several authors have taken an inverse problem approach to the problem (1)-(2) (for Example [7-9]). The novelty of this method lies in the use of nodal points of the eigenfunctions of (1) as spectral parameters. In later years, inverse nodal problems were studied by several authors (see [10-12]). In this work, we obtain an asymptotic formula for the nodal points, and prove uniqueness of the function $q_1(x), h, H$ by a dense set of nodal points, the function $q_0(x)$.

## 2 Asymptotic of the eigenvalues and eigenfunctions

In this section, we investigate the oscillation of the eigenfunctions corresponding to large modulus eigenvalues is established. Let $\varphi(x,\rho), C(x,\rho), S(x,\rho)$ be the solutions of

* **Corresponding Author. (✉)**
**E-mail:** sh.akbarpour@jouybariau.ac.ir

equation (1) under initial conditions $C(0,\rho)=S'(0,\rho)=\varphi(0,\rho)=1,\quad C'(0,\rho)=S(0,\rho)=0, \varphi'(0,\rho)=h$. Then $U(\varphi)=0$. The eigenvalues of $L$ coincide with the zeros of its characteristic function $\Delta(\rho)=-V(\varphi)$. Denote

$$Q(x)=\int_0^x q_1(t)dt, \qquad \int_0^1 q_1(t)dt=0. \tag{3}$$

Using the standard approach (see, e.g., [13]) one can establish the asymptotics

$$S(x,\rho)=\frac{\sin(\rho x-q(x))}{\rho}+\xi(x,\rho), \tag{4}$$

where

$$\xi^{(\nu)}(x,\rho)=O(\frac{1}{\rho^{2-\nu}}e^{|\tau|x}), \qquad \nu=0,1, \qquad |\rho|\to\infty,$$

and

$$C(x,\rho)=\cos(\rho x-Q(x))+\frac{\sin(\rho x-Q(x))}{2\rho}\int_0^x q_1(t)dt+\eta(x,\rho), \tag{5}$$

where

$$\eta^{(\nu)}(x,\rho)=O(\frac{1}{\rho^{2-\nu}}e^{|\tau|x}), \qquad \nu=0,1, \qquad |\rho|\to\infty,$$

uniformly with respect to $x\in[0,1]$. Since $\varphi(x,\rho)=C(x,\rho)+hS(x,\rho)$, for $|\rho|\to\infty$ uniformly in $x$ and using (4), (5) one has (see [14] or chapter 1 in [15] ) :

$$\varphi(x,\rho)=\cos(\rho x-Q(x))+\frac{\sin(\rho x-Q(x))}{\rho}(h+\frac{1}{2}\int_0^x q_1(t)dt)+O(\frac{1}{\rho^2}e^{|\tau|x}), \tag{6}$$

and

$$\varphi'(x,\rho)=(-\rho+q_1(x))\sin(\rho x-Q(x))+(h+\frac{1}{2}\int_0^x q_1(t)dt)\cos(\rho x-Q(x))+O(\frac{1}{\rho}e^{|\tau|x}).$$

(7)

It follows from (6), (7) that for $|\rho|\to\infty$,

$$\Delta(\rho)=(\rho-q_1(1))\sin\rho-(h+H)\cos\rho+O(\frac{1}{\rho}e^{|\tau|}). \tag{8}$$

Using (8) by the well-known method (see, for example, [13]) one has that for $|n|\to\infty$,

$$\rho_n=n\pi+\frac{h+H}{n\pi-q_1(1)}+O(\frac{1}{n^2}). \tag{9}$$

The eigenfunctions of the boundary problem $L$ have the form $y_n(x)=\varphi(x,\lambda_n)$. Substituting (9) into (6) we obtain the following asymptotic formulae for $|n|\to\infty$ uniformly in $x$ (see [16]):

$$y_n(x) = \cos(n\pi x - Q(x)) - \sin(n\pi x - Q(x))\frac{h+H}{n\pi - q_1(1)}$$
$$+\frac{h+\frac{1}{2}Q(x)}{n\pi}\sin(n\pi x - Q(x)) + O(\frac{1}{n^2}). \tag{10}$$

## 3 Computation the nodal points. Inverse problem

For the boundary value problem $L$ an analog of Sturm's oscillation theorem is true. More precisely, the eigenfunction $y_n(x)$ has exactly $n$ zeros inside the interval $(0,1)$. Namely, $0 < x_n^1 < ... < x_n^n < 1$. Denote $\alpha_n^j = \frac{(j-\frac{1}{n})}{n}$. Taking (10) into account, we obtain the following asymptotic formulae for nodal points as $|n| \to \infty$ uniformly in $j$:

$$x_n^j = \alpha_n^j + \frac{1}{n\pi}[-\frac{h+H}{n\pi - q_1(1)}x_n^j + \frac{h+\frac{1}{2}Q(x_n^j)}{n\pi} + Q(x_n^j)] + O(\frac{1}{n^3}). \tag{11}$$

Using this formulae, we arrive at the following assertion.

**Theorem 1.** Fix $x \in [0,1]$. Choose $j_n$ such that $x_n^j \to x$ as $n \to \infty$. Then there exists finite limits and the corresponding equalities hold:

$$Q(x) = \pi \lim_{n\to\infty}[nx_n^{j_n} - (j_n - \frac{1}{2})], \tag{12}$$

$$f(x) = \pi \lim_{n\to\infty}[n(\pi(nx_n^{j_n} - (j_n - \frac{1}{2})) - Q(x_n^{j_n}))], \tag{13}$$

and

$$f(x) = -(h+H)x + h + \frac{1}{2}Q(x). \tag{14}$$

Let us now formulate a uniqueness theorem.

**Theorem 2.** The specification of any dense subset $X_0 \subset X$ uniquely determines the function $q_1(x)$ on $(0,1)$ and the coefficients of the boundary conditions. The function $q_1(x)$ and numbers $h, H$ can be constructed via the formulae

$$q_1(x) = Q'(x), \tag{15}$$

$$h = f(0), \qquad H = -f'(x) - h + \frac{1}{2}q_1(x), \tag{16}$$

where $f(x)$ is calculated by (13).

**proof.** Let a dense subset $X_0$ of the nodal points be given. Then for each $x \in [0,1]$ choose a sequence $X_n^{j_n} \subset X_0$ such that $x_n^{j_n} \to x$ as $|n| \to \infty$. Find the function $Q(x)$ via (12) and formulate (15)-(16) follow from (3) and (14). Note that if $X = \tilde{X}$ , then (12) and (13) yield $Q(x) = \tilde{Q}(x)$ and $f(x) = \tilde{f}(x)$, $x \in [0,1]$. By virtue of (15)-(16), we get $q_1(x) = \tilde{q}_1(x)$ a.e. on $[0,1]$, $h = \tilde{h}, H = \tilde{H}$ **.**

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